GUESSING ABOUT GUESSING: PRACTICAL STRATEGIES FOR CARD GUESSING WITH FEEDBACK

By

Persi Diaconis
Ron Graham
Sam Spiro

Technical Report No. 2020-18
December 2020

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065

GUESSING ABOUT GUESSING: PRACTICAL STRATEGIES FOR CARD GUESSING WITH FEEDBACK

By

Persi Diaconis
Stanford University

Ron Graham
Sam Spiro
University of California San Diego

Technical Report No. 2020-18
December 2020

This research was supported in part by National Science Foundation grant DMS 1954042.

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://statistics.stanford.edu

Guessing about Guessing: Practical Strategies for Card Guessing with Feedback

Persi Diaconis * Ron Graham† Sam Spiro‡

December 7, 2020

Abstract

In simple card games, cards are dealt one at a time and the player guesses each card sequentially. We study problems where feedback (e.g. correct/incorrect) is given after each guess. For decks with repeated values (as in blackjack where suits do not matter) the optimal strategy differs from the "greedy strategy" (of guessing a most likely card each round). Further, both optimal and greedy strategies are far too complicated for real time use by human players. Our main results show that simple heuristics perform close to optimal.

1 Introduction

Consider the following game. A deck of n cards labeled $1, 2, \ldots, n$ is randomly shuffled. A player then guesses each card (sequentially) as the cards are dealt face down on the table. After each guess, the player is given some amount of feedback. The main cases that we consider are the following:

- No feedback (the player is told nothing after each guess),
- Complete feedback (the player is shown the value of the card after each guess),
- Yes/No feedback (the player is only told whether their guess was correct or not).

The final score is the total number of correct guesses made after all n cards have been drawn. Suppose (for now) that the player uses some fixed strategy which maximizes their score under the given level of feedback. Let $\mathbf{X}_i = 1$ if the ith guess is correct (under this fixed strategy) and $\mathbf{X}_i = 0$ otherwise. Thus $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$ is the score in the game and:

*Dept. of Mathematics and Statistics, Stanford University. Research supported by NSF Grant DMS1954042.
†Dept. of Computer Science and Engineering, UCSD, graham@ucsd.edu.
‡Dept. of Mathematics, UCSD sspiro@ucsd.edu. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1650112.

- With No feedback, $\Pr[\mathbf{X}_i = 1] = n^{-1}$ for all $1 \le i \le n$, so the expected number of correct guesses is
$$\mathbb{E}[\mathbf{S}_n] = n \cdot n^{-1} = 1.$$

- With Complete feedback, the maximizing strategy is to always guess a card known to be left in the deck. Thus
$$\Pr[\mathbf{X}_1 = 1] = \frac{1}{n},\ \Pr[\mathbf{X}_2 = 1] = \frac{1}{n-1}, \ldots,\ \Pr[\mathbf{X}_i = 1] = \frac{1}{n-i+1}, \ldots,$$
so $\mathbb{E}[\mathbf{S}_n] = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + O(1)$.

- With Yes/No feedback, consider the following strategy. Guess '1'. If correct, then the player knows '1' is no longer in the deck and they will never guess it again. If incorrect, all the player knows for sure is that '1' is still in the deck. Thus, heuristically, a good strategy would be to guess '1' until the player is told they are correct, then '2' until they are told they are correct (or until the deck runs out), then '3', and so on. And indeed, this strategy turns out to be optimal, see Diaconis and Graham [7].

  Under this strategy, the player always gets at least 1 correct guess, that is, $\Pr[\mathbf{S}_n \ge 1] = 1$. There are at least 2 correct guesses if and only if '2' appears after '1', so $\Pr[\mathbf{S}_n \ge 2] = \frac{1}{2}$. More generally, for all $1 \le k \le n$ we have $\Pr[\mathbf{S}_n \ge k] = \frac{1}{k!}$. Thus under optimal play,
$$\mathbb{E}[\mathbf{S}_n] = \sum_{k=1}^{n} \Pr[\mathbf{S}_n \ge k] = 1 + \frac{1}{2} + \cdots + \frac{1}{n!} = e - 1 + O(1/n!).$$

In real card games, cards have repeated values. For example, in Blackjack or Baccarat, suits do not matter and all of the card types '10,J,Q,K' have value 10. In classical parapsychology experiments, a deck of 25 cards are used with five copies of five different card types.

The analysis for No feedback and Complete feedback in the repeated setting is quite similar to the case when there are no repeated card types, but for Yes/No feedback things become more complex. In particular, Diaconis and Graham [7] posed the following simple problems when the deck has $2n$ cards with two copies of each card type labeled $1, \ldots, n$.

- In this setting, is the optimal score unbounded as a function of n?

- It turns out that the greedy strategy of guessing a most likely card type each round is *not* an optimal strategy in this setting. Further, the optimal strategy and the greedy strategy seem impossibly complicated to implement by human players in practice. Are there simpler strategies that perform reasonably well?

The answer to this first question has recently been answered negatively by Diaconis, Graham, He, and Spiro [9]. In this paper we focus on the second question. More precisely, we consider decks with mn cards where each of the n card types appear m times[1]. We define $\mathbf{S}_{m,n}$ to be the number of correct guesses made by the player with this deck if they use a given strategy

[1] A useful mnemonic is that m is the multiplicity of each card type, while n is the number of card types.

under some level of feedback. Our goal is to use practical strategies to bound the maximum and minimum possible values of $\mathbb{E}[\mathbf{S}_{m,n}]$ under Yes/No feedback.

To this end, we say that a strategy is an *optimal strategy* if it achieves $\max \mathbb{E}[\mathbf{S}_{m,n}]$, where the maximum ranges over all strategies, and similarly we say a strategy is an *optimal misère strategy* if it achieves $\min \mathbb{E}[\mathbf{S}_{m,n}]$. With this in mind, we state our main results of the paper.

Theorem 1.1. *For all $m \geq 1$ and $n \geq 64m$:*

- *Under the optimal strategy with Yes/No feedback,*

$$\mathbb{E}[\mathbf{S}_{m,n}] \geq m + .01\sqrt{m}.$$

- *Under the optimal misère strategy with Yes/No feedback,*

$$\mathbb{E}[\mathbf{S}_{m,n}] \leq m - .01\sqrt{m}.$$

It was proven in [9] that $\mathbb{E}[\mathbf{S}_{m,n}] \leq m + O(m^{3/4} \log m)$ under the optimal strategy and n sufficiently large in terms of m, so one can not prove a lower bound that is much stronger than Theorem 1.1 for m large (though the constant .01 can easily be improved). We also obtain bounds that are reasonable for small m.

Theorem 1.2. *For n sufficiently large, under the optimal strategy with Yes/No feedback,*

$$\mathbb{E}[\mathbf{S}_{2,n}] \geq 2.91,$$

$$\mathbb{E}[\mathbf{S}_{3,n}] \geq 3.97.$$

Computational evidence suggests that the actual value of $\mathbb{E}[\mathbf{S}_{2,n}]$ is close to the bound in Theorem 1.2, see Table 3.

Theorem 1.3. *For any fixed m and n sufficiently large in terms of n, under the optimal misère strategy with Yes/No feedback we have*

$$1 - e^{-m} - o(1) \leq \mathbb{E}[\mathbf{S}_{m,n}] \leq m - 1 + m^{-1} - m^{-1}e^{-m} + o(1).$$

We note that Theorem 1.3 generalize bounds of [7] when $m = 1$ (where the bounds are tight). We also note that for large m the upper bound of Theorem 1.1 is more effective than that of Theorem 1.3.

We hope that our paper contributes to a central problem of artificial intelligence: how to make black box alogorithms “explainable” or “interpetable,” see [16, 18].

Organization and Notation. The rest of the paper is organized as follows. In Section 2 we discuss the history of the problem in more depth. In Section 3 we state some practical strategies that one can implement in the Yes/No feedback model and give some computational data for how they perform. We then prove rigorous bounds on some of these strategies (or slight technical variants thereof) in order to prove our main results. In particular, Theorem 1.1 is proven in Section 4, Theorem 1.2 in Section 5, and Theorem 1.3 in Section 6. Concluding remarks and further problems are given in Section 7.

We gather some notation that we use throughout the text. We use standard asymptotic notation throughout this paper. In particular, if f is a function depending on n, we write $O(f)$ to refer to a function g such that $\limsup_{n\to\infty} \frac{g(n)}{f(n)} < \infty$, and similarly $o(f)$ refers to a function g such that $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$. We write $\Omega(f)$ if $\liminf_{n\to\infty} \frac{g(n)}{f(n)} > 0$ and $\Theta(f)$ to mean a function which is both $O(f)$ and $\Omega(f)$. We write, for example, $O_\epsilon(f)$ to refer to a function such that $\limsup_{n\to\infty} \frac{g(n)}{f(n)}$ is a constant depending on ϵ. Finally, we write $f \sim g$ if $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 1$. For more on asymptotic notation, see [14].

We let $\mathfrak{S}_{m,n}$ refer to the set of all words π consisting of m copies of each element in $[n] := \{1, 2, \ldots, n\}$, which we think of as a permutation of a deck on mn cards with n card types each occurring with multiplicity m. We always write $\boldsymbol{\pi}$ to indicate an element of $\mathfrak{S}_{m,n}$ chosen uniformly at random. If $\mathcal{G}$ is a deterministic strategy for the player under some level of feedback, we let $S(\mathcal{G}, \pi)$ be the score of the player if they follow strategy $\mathcal{G}$ and the deck is shuffled according to π. Thus the expected score following strategy $\mathcal{G}$ is $\mathbb{E}[S(\mathcal{G}, \boldsymbol{\pi})]$. As much as possible we denote random variables, such as $\mathbf{S}_{m,n}$, in bold and strategies, such as $\mathcal{G}$, in script.

2 History

2.1 Real World Applications

Much of the motivation for studying guessing problems which use sampling without replacement and different levels of feedback originates with certain real world problems.

Consider R. A. Fisher's "Lady Tasting Tea" experiment. A lady claims she can tell if the tea in her drink was poured before or after the milk was poured. To test the claim, Fisher proposed the following experiment. Eight cups of tea are randomly placed with four poured before the milk and four after. To help calibrate the lady's response, after each of her guesses, she is told "correct" or "incorrect." If the lady guesses arbitrarily each round, then we expect 4 correct guesses. However, if she uses the optimal strategy (assuming she has no ability to discern the teas), then she can expect to get around 5.3 correct guesses. Conversely, if she's trying to show "there's nothing to it", she might use the optimal misère strategy, giving 3.7 correct guesses in expectation.

Mathematically this is the same problem discussed by Blackwell and Hodges [1], Efron [11], and many later authors in connection with clinical trials. In comparing two treatments on $2m$ patients, suppose it is decided that m patients are to be randomly selected to receive each treatment. Assume the patients arrive sequentially and must be ruled either ineligible or eligible (and then assigned to one of the two treatments). A physician observing the outcome of each trial would know which treatment was most likely used for each trial. This information might bias results if the physician ruled less healthy patients ineligible on trials where a favored treatment was less likely. A natural measure of selection bias is the number of times the physician correctly guesses which treatment will be used next. Blackwell and Hodges [1] showed that $m + \frac{1}{2}\sqrt{\pi m} - \frac{1}{2} + O(1/m)$ correct guesses are made under optimal guessing.

Independently, the same problem is stuided by Ethier and Levin [13] as part of their work in evaluating card counting strategies in casino games such as Blackjack, Baccarat, and Trente

No feedback	Yes/No feedback	Complete feedback
5	6.65	8.65

Table 1: Expected number of correct guesses under an optimal strategy with Yes/No feedback when $m = n = 5$. The results are rounded at two decimal places.

et Quarante. As cards are turned up during play, the deck changes composition, so betting levels and actual strategy changes can make some games favorable. The exact same formula as Blackwell and Hodges appear in Ethier [12, Problem 11.15]. This wonderful book develops many further ideas tailored to these applications.

Finally, consider the problem of testing if a subject has extrasensory perception (ESP). A huge number of trials of the following experiment were performed by J. B. Rhine at the Durham parapsychology lab. A deck of 25 cards with five copies of five different symbols (0, $+$, $\int\int\int$, $\Box$, $*$) is shuffled, and a guessing subject guesses them sequentially as above. It is common practice to give various kinds of feedback in these experiments, which is sometimes inadvertent if the guessing and sending subject are in the same room. For extensive references, see Diaconis [5].

2.2 Some Numbers

The main results in our paper are asymptotic, giving results as m and/or n tend to infinity. It is of interest to obtain exact results for decks of small size, both to get a feel for what the correct asymptotic bounds should be, and to make use of these results in the real world applications mentioned above.

For example, consider the classical ESP experiment which corresponds to $m = n = 5$. In [7], a direct recursion was used to get exact answers record in Table 1. As noted in [5], for actual ESP experiments, the highest recorded scores essentially never exceeded 8.65, and the highest scores among long term subjects never exceeded 6.63; so the numbers of Table 1 help benchmark high scoring experiments. Much more can be said about this well studied case, see Diaconis, Gatto, and Graham [6] and Gatto [15].

The following Table 2 gives some numbers for $m = 2$ and small n under Yes/No feedback using various strategies. The *linear strategy* mentioned in this table is the following: if at any stage it is known that the remaining deck has c_i values of i that have not been guessed correctly and p_i past incorrect guesses of i, then guess the type i^* such that $c_i + .51p_i$ is maximized (breaking ties at random). Table 2 shows that, in addition to the greedy strategy being non-optimal, it is seemingly outperformed by this simple linear strategy.

It is natural to ask "why" the greedy strategy is not optimal with Yes/No feedback. One heuristic is that there is a tradeoff between score and information: sometimes it is better to guess a card which is less likely if knowledge about this card type is more valuable. However, the real answer to this question is "we don't know!"

We find it rather surprising that the simple linear "card counting" strategy performs so well, at least when $m = 2$ and n is small. In the classical ESP setting of $m = n = 5$, a similar strategy can be used where one guesses the i^* which maximizes $c_i + .35p_i$. This simple strategy

	Optimal	Greedy	Linear
$\mathbb{E}[\mathbf{S}_{2,2}]$	2.8333	2.8333	2.8333
$\mathbb{E}[\mathbf{S}_{2,3}]$	3.0111	3.0111	3.0111
$\mathbb{E}[\mathbf{S}_{2,4}]$	3.0452	3.0333	3.0433
$\mathbb{E}[\mathbf{S}_{2,5}]$	3.0467	3.0222	3.0441

Table 2: Expected number of correct guesses with Yes/No feedback under various strategies when $m = 2$. The results are rounded at four decimal places.

n	6	7	8	9	10
$\mathbb{E}[\mathbf{S}_{2,n}]$	3.0376	3.0323	3.0260	3.0219	3.0186

Table 3: Expected number of correct guesses with Yes/No feedback under an optimal strategy when $m = 2$. The results are rounded at four decimal places.

has expectation 6.6149, which is better than the greedy strategy! Here .35 is the result of a computer search, which we think of as "three and a half wrongs make one right."

Observe by Table 2 that the greedy and linear strategy are close to optimal for small n, though we are unable to prove that this holds in general. At a first glance of Table 2, it may appear as if $\mathbb{E}[\mathbf{S}_{2,n}]$ under the optimal strategy is non-decreasing, but Table 3 shows that this is not the case. Note that all of these values seem to be very close to 3, but it is unclear what the true asymptotic value should be. Note that Theorem 1.2 shows that this asymptotic value is as least 2.91.

2.3 Previous Research

This paper continues research in [7, 9]. The following result from these papers essentially solves the problem for Complete feedback, where here and throughout log denotes the natural logarithm.

Theorem 2.1 ([7, 9])**.** *The strategy $\mathcal{G}^+$ (resp. $\mathcal{G}^-$) of guessing a most (resp. least) likely card at each step is an optimal strategy (resp. optimal misère strategy) under Complete feedback. Moreover, under strategy $\mathcal{G}^\pm$ we have*

$$\mathbb{E}[\mathbf{S}_{m,n}] = m \pm M_n\sqrt{m} + o_n(\sqrt{m}),$$

where $M_n = \Theta(\sqrt{\log n})$ is the expected maximum value of n independent standard normal variables. We also have under $\mathcal{G}^+$ that

$$\mathbb{E}[\mathbf{S}_{m,n}] = H_m \log n + o_m(\log n),$$

where $H_m = \sum_{i=1}^m i^{-1}$ is the m-th harmonic number, and under $\mathcal{G}^-$ that

$$\mathbb{E}[\mathbf{S}_{m,n}] = \Theta_m(n^{-1/m}),$$

Despite not knowing the optimal strategy under Yes/No feedback in general, it was proven in [9] that asymptotically $\mathbb{E}[\mathbf{S}_{m,n}]$ is at most m.

Theorem 2.2 ([9])**.** *If n is sufficiently large in terms of m, then under an optimal strategy with Yes/No feedback,*

$$\mathbb{E}[\mathbf{S}_{m,n}] = m + O(m^{3/4}\log^{1/4} m).$$

This shows that the lower bound of Theorem 1.1 is close to best possible. It also shows that for fixed m, under an optimal strategy with Yes/No feedback, $\mathbb{E}[\mathbf{S}_{m,n}]$ is bounded as a function of n. This is in sharp contrast to the Complete feedback model, where one could guarantee $\Omega(\log n)$ correct guesses in expectation.

Theorem 2.2 does not given an effective bound on $\mathbb{E}[\mathbf{S}_{m,n}]$ under Yes/No feedback for any fixed m, and it is natural to ask what happens for fixed m. The case for $m = 1$ was completely solved in [7], see also the discussion after Theorem 1.3.

Theorem 2.3 ([7])**.** *Under the optimal strategy with Yes/No feedback,*

$$\mathbb{E}[\mathbf{S}_{m,n}] = e - 1 + O(1/n!) \approx 1.72,$$

and under the optimal misère strategy,

$$\mathbb{E}[\mathbf{S}_{m,n}] = 1 - e^{-1} + O(1/n!) \approx .632.$$

The proof of Theorem 2.3 relied on an optimal strategy which is easy to analyze when $m = 1$, and without this it seems difficult to nail down asymptotic values of $\mathbb{E}[\mathbf{S}_{m,n}]$ in general.

To analyze $\mathbf{S}_{m,n}$ under Yes/No feedback, it is often useful to determine the probability that the next card is of type i given the past history of feedback. In Chung, Diaconis, Graham, and Mallows [2], this problem is reduced to evaluating the permanent of certain zero/one matrices. While general evaluation of permanents is #P-complete, the matrices arising here have enough structure to permit evaluation in practical problems. The study of permanents in relation to card guessing problems is further studied in [3, 8, 10].

Lastly, we note that in real life applications, the deck is not always shuffled uniformly at random, and it is natural to consider what happens with other common forms of shuffling. The case where dovetail shuffles are used is studied by Ciucu [4], and recently Liu [17] investigated the case where riffle shuffles are used. We emphasize that for our results, we only consider decks which are shuffled uniformly at random.

3 Practical Strategies and Computational Data

Perhaps the simplest strategy to use in the Yes/No feedback model is the strategy where one guesses '1' every round. This strategy always gives exactly m correct guesses. Of course, this strategy is somewhat silly. One can always do at least as well by using the strategy: guess card '1' until the player is told m guesses were correct, then guess card '2' until the player is told m guesses were correct, and so on. We call this the *safe strategy*.

With this we have $\mathbf{S}_{m,n} \geq m$ regardless of how the deck is shuffled. However, we have $\Pr[\mathbf{S}_{m,n} = m] = \frac{1}{2}$, as half the time the last '1' appears after the last '2'. Similarly one can show that $\Pr[\mathbf{S}_{m,n} \geq 2m] = \binom{2m}{m}^{-1} \sim \sqrt{\pi m} 4^{-m}$, because this only happens if all of the '1's appear before all of the '2's. Using similar reasoning, it is not too difficult to convince oneself that under the safe strategy, $\mathbb{E}[\mathbf{S}_{m,n}] = m + \Theta(\sqrt{m})$ when n is sufficiently large in terms of m.

Another natural strategy is the *shifting strategy* $\mathcal{F}$ where the player guesses '1' until they get a correct guess, then '2' until they get a correct guess, and so on; and upon guessing an 'n' correctly, they go back to guessing '1's, and then '2's, and so on.

Observe that the safe and shifting strategies look identical until the first '1' is guessed correctly. Because of this, the player can choose which strategy they wish to use after seeing the first '1', where intuitively they should use the shifting strategy if the first '1' shows up early and the safe strategy otherwise. To this end, we define the *γ-shifting strategy* $\mathcal{F}_\gamma$ by guessing '1' until a correct guess is made at time t. If $t \geq \gamma mn$, the player proceeds as in the safe strategy, and otherwise they proceed as in the shifting strategy. For example, $\mathcal{F}_0$ is the safe strategy and $\mathcal{F}_1$ is the shifting strategy, so $\mathcal{F}_\gamma$ serves as a sort of interpolation between these two strategies.

Of course one can generalize this idea even further. For example, one can define the *(γ, γ')-shifting strategy* $\mathcal{F}_{\gamma,\gamma'}$ by guessing '1' until a correct guess is made at time t. If $t \geq \gamma mn$ then one plays as in the safe strategy, otherwise one starts playing as in the shifting strategy until a '2' is guessed correctly at time t'. If $t' \geq \gamma' mn$, then one essentially switches to the safe strategy (continuing to guess '2's until all of them are guessed then '3's and so on), and otherwise one keeps using the shifting strategy. For example, $\mathcal{F}_{\gamma,1}$ is the same as $\mathcal{F}_\gamma$. One can continue to build upon this idea with an arbitrary amount of complexity.

Another simple strategy is the *halfway strategy* $\mathcal{H}^+$ which is defined by guessing '1' for the first $\frac{1}{2}mn$ trials. If at most $\frac{1}{2}m$ cards have been guessed correctly, then keep guessing '1' (guaranteeing m points at the end of the game), and otherwise guess '2' for the rest of the game. The intuition is that if there are many copies of '1' in the first half of the deck, then there will be slightly more copies of '2' (or any other card type) in the second half of the deck.

We next turn to strategies for getting few correct guesses. Analogous to $\mathcal{H}^+$, we define the halfway strategy $\mathcal{H}^-$ by guessing '1' for the first half of the game, then to keep guessing '1's if more than $\frac{1}{2}m$ correct guesses were made, and otherwise guessing '2's the rest of the game. Finally, we define the *avoiding strategy* $\mathcal{A}$ by guessing '1', then '2', then '3', and so on until some 'k' is guessed correctly, at which point one guesses 'k' for the rest of the game. If the player does not guess any of the first n cards correctly, then they again guess '1', then '2', and so on until a card 'k' is guessed correctly, and then they guess 'k' for the rest of the game.

In Table 4, we present computational data for most of these strategies. The first two rows are the exact value of $\mathbb{E}[\mathbf{S}_{m,n}]$ using the stated strategy with Yes/No feedback. The next two rows represent the sample mean of $S(\mathcal{G}, \boldsymbol{\pi})$ obtained after sampling $t = 10^6$ permutations of $\mathfrak{S}_{m,n}$. The entries corresponding to $\mathcal{F}_\gamma$ and $\mathcal{F}_{\gamma,\gamma'}$ indicate which values of γ, γ' are used, with these values chosen to roughly maximize the expectation. Each entry is rounded after three decimal places.

We note that for $m = 2$, all of these strategies perform worse than the simple linear strategy defined prior to Table 2. The main benefit of the strategies of this section is that we can give

	$\mathcal{S}$	$\mathcal{F}$	$\mathcal{F}_\gamma$	$\mathcal{F}_{\gamma,\gamma'}$	$\mathcal{H}^+$	$\mathcal{H}^-$
$m = 2, n = 6$ (exact)	2.737	2.751	$\mathcal{F}_{.3}$: 2.941	$\mathcal{F}_{.3,.5}$: 2.990	2.212	1.682
$m = 3, n = 5$ (exact)	3.772	3.753	$\mathcal{F}_{.25}$: 4.006	$\mathcal{F}_{.25,.4}$: 4.084	3.431	2.569
$m = 4, n = 10$ ($t = 10^6$)	4.806	4.831	$\mathcal{F}_{.2}$: 5.100	$\mathcal{F}_{.2,.25}$: 5.172	4.370	3.585
$m = 5, n = 20$ ($t = 10^6$)	5.835	5.856	$\mathcal{F}_{.15}$: 6.136	$\mathcal{F}_{.15,.25}$: 6.210	5.482	4.516

Table 4: (Simulated) values of $\mathbb{E}[\mathbf{S}_{m,n}]$ under various strategies.

rigorous bounds on their performance (or more precisely, on technical variants of the strategies which are easier to analyze). For example, we provided no computational data for the avoiding strategy, since it turns out that we can determine its expectation asymptotically for all m.

Proposition 3.1. *For any fixed m, using the avoiding strategy with Yes/No feedback gives*

$$\mathbb{E}[\mathbf{S}_{m,n}] \sim m - 1 + m^{-1} - m^{-1}e^{-m}.$$

4 Proof of Theorem 1.1: the Halfway Strategies $\mathcal{H}^{\pm}$

In this section we prove Theorem 1.1 using a technical variant of the halfway strategies $\mathcal{H}^{\pm}$, which we denote by $\mathcal{H}^{\pm}_*$. We define $\mathcal{H}^{+}_*$ by guessing '1' a total of $\lfloor mn/2 \rfloor$ times, with us continuing to guess '1' if at most $\frac{1}{2}m + \frac{1}{2}\sqrt{m}$ correct guesses have been made, and with '2' being guessed the rest of the game otherwise. Similarly $\mathcal{H}^{-}_*$ is defined by guessing '1' $\lfloor mn/2 \rfloor$ times, then continuing if at most $\frac{1}{2}m - \frac{1}{2}\sqrt{m}$ correct guesses have been made and guessing '2' otherwise.

To show that $\mathcal{H}^{+}_*$ gives the correct lower bound, we first show that the probability of getting at least $\frac{1}{2}m + \frac{1}{2}\sqrt{m}$ correct guesses in the first phase is relatively large.

Lemma 4.1. *For $\pi \in \mathfrak{S}_{m,n}$, let $K_1(\pi)$ denote the number of $t \le \lfloor mn/2 \rfloor$ with $\pi_t = 1$. Then for $n \ge 64m$ and all $0 \le k \le m$, we have*

$$\Pr[K_1(\boldsymbol{\pi}) = k] \ge \frac{2}{3} \cdot 2^{-m} \binom{m}{k}$$

That is, the probability of having exactly k of the m '1's appear in the first half of $\boldsymbol{\pi}$ is roughly the probability of having k heads in a series of m coin tosses.

Proof. First assume mn is even. Then

$$\Pr[K_1(\boldsymbol{\pi}) = k] = \binom{mn/2}{k}\binom{mn/2}{m-k} \Big/ \binom{mn}{m}. \tag{1}$$

We recall the bounds $\binom{N}{r} \le N^r/r!$ and

$$\binom{N}{r} \ge \frac{(N-r)^r}{r!} = (1 - r/N)^r \frac{N^r}{r!} \ge (1 - r^2/N)\frac{N^r}{r!},$$

where the last inequality follows from $(1+x)^r \geq 1 + rx$, which is valid for $x \geq -1$ and $r \geq 1$. Using these bounds and (1), we find for $0 \leq k \leq m$ and $n \geq 64m$ that

$$\begin{aligned}\Pr[K_1(\boldsymbol{\pi}) = k] &\geq \frac{(mn/2)^m m!}{(mn)^m k!(m-k)!}(1 - 2k^2/mn)(1 - 2(m-k)^2/mn) \\ &\geq 2^{-m}\binom{m}{k}(1 - 2m/n)^2 \geq 2^{-m}\binom{m}{k} \cdot \frac{2}{3}.\end{aligned}$$

The completes the proof for mn even. If mn is odd, note that $mn/2 - 1 \geq mn/4$ for $n \geq 4$. With this and a similar analysis as before, we find

$$\begin{aligned}\Pr[K(\boldsymbol{\pi}) = k] &\geq 2^{-m}\binom{m}{k}\frac{(mn-1)^k(mn+1)^{m-k}}{(mn)^m}(1 - m/(mn/2 - 1))^2 \\ &\geq 2^{-m}\binom{m}{k}(1 - 1/mn)^m(1 - 4m/n)^2 \geq 2^{-m}\binom{m}{k}(1 - 4m/n)^3.\end{aligned}$$

Using $n \geq 64m$ gives the desired result.

□

We next show that, conditional on getting at least $\frac{1}{2}m + \frac{1}{2}\sqrt{m}$ correct guesses in the first phase of the strategy, one expects to guess '2' correctly at least $\frac{1}{2}m$ times in the second phase.

Lemma 4.2. *For $\pi \in \mathfrak{S}_{m,n}$, let $K_2(\pi)$ denote the number of $t > \lfloor mn/2 \rfloor$ with $\pi_t = 2$, and define $K_1(\pi)$ as in Lemma 4.1. If $k \geq \frac{1}{2}m$, then*

$$\mathbb{E}[K_2(\boldsymbol{\pi})|K_1(\boldsymbol{\pi}) = k] \geq \frac{1}{2}m,$$

and if $k \leq \frac{1}{2}m - 1$,

$$\mathbb{E}[K_2(\boldsymbol{\pi})|K_1(\boldsymbol{\pi}) = k] \leq \frac{1}{2}m,$$

Proof. Let π be such that $K_1(\pi) = k$. Let $t_1 < \cdots < t_{m(n-1)}$ be the set of indices with $\pi_{t_j} \neq 1$ for all j. For $0 \leq s \leq m(n-1) - 1$, define $\pi^{(s)} \in \mathfrak{S}_{m,n}$ by $\pi^{(s)}_t = \pi_t$ if $\pi_t = 1$ (i.e. if $t \neq t_j$ for any j), and otherwise define $\pi^{(s)}_{t_j} = \pi_{t_{j-s}}$, where we define $t_{j-s} := t_{m(n-1)+j-s}$ whenever $j - s \leq 0$. That is, $\pi^{(s)}$ is π after cyclically shifting the non-1 entries s indices to the left. In particular, $K_1(\pi^{(s)}) = K_1(\pi) = k$ for all s. Moreover, for each $\sigma \in \mathfrak{S}_{m,n}$ with $K_1(\sigma) = k$, there exist exactly $m(n-1)$ pairs (π, s) with $\pi^{(s)} = \sigma$ and $K_1(\pi) = k$; namely, these are the pairs of the form $(\sigma^{(s')}, m(n-1) - s')$. Thus

$$\sum_{\pi : K_1(\pi) = k} K_2(\pi) = \frac{1}{m(n-1)} \sum_{\pi : K_1(\pi) = k} \sum_s K_2(\pi^{(s)}).$$

Dividing this expression by the number of π with $K_1(\pi) = k$ gives

$$\mathbb{E}[K_2(\boldsymbol{\pi})|K_1(\boldsymbol{\pi}) = k] = \frac{1}{m(n-1)}\mathbb{E}\left[\sum_s K_2\left(\boldsymbol{\pi}^{(s)}\right) \mid K_1(\boldsymbol{\pi}) = k\right]. \tag{2}$$

We claim for all π with $K_1(\pi) = k$ that

$$\sum_s K_2(\pi^{(s)}) = (mn - \lfloor mn/2 \rfloor + k - m) \cdot m.$$

Indeed, observe that $t_j > \lfloor mn/2 \rfloor$ for exactly $mn - \lfloor mn/2 \rfloor + k - m$ values of j (this holds when $k = m$, and each decrease in k decreases this count by 1). Further, for each j we have $\pi^{(s)}_{t_j} = 2$ for exactly m choices of s. The claim then follows from these two observations.

With this claim and (2), if $k \ge \frac{1}{2}m$ we find

$$\mathbb{E}[K_2(\boldsymbol{\pi})|K_1(\boldsymbol{\pi}) = k] \ge \frac{1}{m(n-1)}(mn - mn/2 - m/2)m = \frac{1}{2}m,$$

and similarly if $k \le \frac{1}{2}m - 1$ we have

$$\mathbb{E}[K_2(\boldsymbol{\pi})|K_1(\boldsymbol{\pi}) = k] \le \frac{1}{m(n-1)}(mn - (mn/2 - 1) - m/2 - 1)m = \frac{1}{2}m,$$

proving the result. □

Lastly, we require an anti-concentration result for binomial random variables.

Lemma 4.3. *Let $\mathbf{B}_m$ denote a binomial random variable with m trials and probability of success 1/2. Then for all m,*

$$\Pr\left[\mathbf{B}_m \ge \frac{1}{2}m + \frac{1}{2}\sqrt{m}\right] \ge .03$$

Proof. Let $\mathbf{X}_i$ be independent random variables with $\Pr[\mathbf{X}_i = \pm 1] = \frac{1}{2}$ for all i, and let $\mathbf{S}_m = \sum_{i=1}^m \mathbf{X}_i$. Because each of the $\mathbf{X}_i$ have zero mean, unit variance, and third absolute moment 1, we can apply a Berry-Essen type inequality due to Shevtsova [19] to conclude for all x, m that

$$|\Pr[\mathbf{S}_m < x\sqrt{m}] - \Phi(x)| \le 1.415m^{-1/2},$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the cumulative distribution function of a standard normal distribution. Observe that $\mathbf{B}_m$ is distributed the same as $\frac{1}{2}(\mathbf{S}_m + m)$, which implies

$$\left|\Pr\left[\mathbf{B}_m < \frac{1}{2}m + \frac{1}{2}x\sqrt{m}\right] - \Phi(x)\right| \le 1.415m^{-1/2}.$$

In particular, plugging $x = 1$ into this expression gives $\Pr[\mathbf{B}_m < \frac{1}{2}m + \frac{1}{2}\sqrt{m}] \le .85 + 1.415m^{-1/2}$. Thus for $m \ge 150$ we have $\Pr[\mathbf{B}_m \ge \frac{1}{2}m + \frac{1}{2}\sqrt{m}] \ge .03$, and one can verify that this inequality holds for all smaller m by aid of a computer. □

Proof of Theorem 1.1. We first prove our lower bound on the optimal strategy by considering the aforementioned strategy $\mathcal{H}_*^+$ of guessing '1' a total of $\lfloor mn/2 \rfloor$ times, then guessing '1' the rest of the game if we guessed fewer than $\frac{1}{2}m + \sqrt{m}$ cards correctly, and otherwise guessing '2' for the rest of the game.

For ease of notation, we let $\mathbf{K}_i = K_i(\boldsymbol{\pi})$ for $i = 1, 2$ as defined in Lemmas 4.1 and 4.2. With this we see that $S(\mathcal{H}_*^+, \boldsymbol{\pi}) = m$ if $\mathbf{K}_1 < \frac{1}{2}m + \frac{1}{2}\sqrt{m}$ and $S(\mathcal{H}_*^+, \boldsymbol{\pi}) = \mathbf{K}_1 + \mathbf{K}_2$ otherwise. Thus

$$\begin{aligned}
\mathbb{E}[S(\mathcal{H}_*^+, \boldsymbol{\pi})] &= \Pr\left[\mathbf{K}_1 < \frac{1}{2}m + \frac{1}{2}\sqrt{m}\right] \cdot m + \sum_{k \geq \frac{1}{2}m + \frac{1}{2}\sqrt{m}} \Pr[\mathbf{K}_1 = k] \cdot (k + \mathbb{E}[\mathbf{K}_2 | \mathbf{K}_1 = k]) \\
&\geq \Pr\left[\mathbf{K}_1 < \frac{1}{2}m + \frac{1}{2}\sqrt{m}\right] \cdot m + \sum_{k \geq \frac{1}{2}m + \frac{1}{2}\sqrt{m}} \Pr[\mathbf{K}_1 = k] \cdot \left(\frac{1}{2}m + \frac{1}{2}\sqrt{m} + \frac{1}{2}m\right) \\
&= m + \Pr\left[\mathbf{K}_1 \geq \frac{1}{2}m + \frac{1}{2}\sqrt{m}\right] \cdot \frac{1}{2}\sqrt{m},
\end{aligned}$$

where the last inequality used Lemma 4.2. By Lemmas 4.1 and 4.3, we have

$$\Pr\left[\mathbf{K}_1 \geq \frac{1}{2}m + \frac{1}{2}\sqrt{m}\right] \geq \frac{2}{3} \Pr\left[\mathbf{B}_m \geq \frac{1}{2}m + \frac{1}{2}\sqrt{m}\right] \geq .02,$$

and with this we conclude the desired lower bound for the optimal strategy.

For the optimal misère strategy, essentially the same analysis as above applies to $\mathcal{H}_*^-$, which gives the desired result. □

5 Proof of Theorem 1.2: the γ-shifting Strategy $\mathcal{F}_\gamma$

Intuitively we use the γ-shifting strategy $\mathcal{F}_\gamma$ to achieve the lower bound of Theorem 1.2 for $\mathbb{E}[S_{2,n}]$ and $\mathbb{E}[S_{3,n}]$, though for technical reasons it will be convenient to use a variant which never guesses card types larger than some cutoff value k. To aid in our proof, we use the following lemma for approximating sums with integrals.

Lemma 5.1 ([14])**.** *Let $a < b$ be integers. Let $h(x)$ be an integrable function in $[a-1, b+1]$, $S = \sum_{i=a}^{b} h(i)$, and $I = \int_a^b h(x)dx$. Let M be such that $|h(x)| \leq M$ for all $a - 1 \leq x \leq b + 1$. Suppose $[a-1, b+1]$ can be broken up into at most r intervals such that $h(x)$ is monotone on each. Then*

$$|S - I| \leq 6rM.$$

Proof of Theorem 1.2. The argument is somewhat complex, so to start we consider $m = 2$. Fix some integer k and real γ, and define the (k, γ)-shifting strategy $\mathcal{F}_{k,\gamma}$ as follows. The player guesses '1' until they get a correct guess. If this happens in less than $2\gamma n$ guesses, they use a modified shifting strategy $\mathcal{F}_k$ where they guess '2' until they get a correct guess, then '3' until they get a correct guess, all the way up to 'k', after which they go back to guessing '1' until they get a correct guess, then '2', and so on. If they guess 'k' correctly a second time, they keep guessing 'k' (effectively giving up on trying to get more points). If the first '1' appears after $2\gamma n$ guesses, they use a modified safe strategy $\mathcal{S}_k$ where they guess '1' until they guess both correctly, then '2' until they guess both correctly, and so on until they guess both 'k's correctly, after which they keep guessing 'k' for the rest of the game.

Given a permutation π, let $F_k(\pi)$ be the score the player gets using the modified shifting strategy $\mathcal{F}_k$ on π, and let $G_k(\pi)$ be the score they get using the modified safe strategy $\mathcal{S}_k$ on

π. Let π_1^{-1} denote the index of the leftmost '1' in π. For example, if $k=3$ and $\pi = 41234132$, then $\pi_1^{-1} = 2$, $F_3(\pi) = 5$ (coming from guessing 1, 2, 3, 1, 2 correctly) and $G_3(\pi) = 3$ (coming from guessing 1, 1, 2 correctly). As another set of examples, $\{(\pi, \pi_1^{-1}, F_2(\pi), G_2(\pi)) : \pi \in \mathfrak{S}_{2,2}\}$ is equal to

$$\{(1122,1,2,4),\ (1212,1,4,3),\ (1221,1,3,2),\ (2112,2,2,3),\ (2121,2,3,2),\ (2211,3,1,2)\}. \tag{3}$$

Let $\pi^{\leq k}$ denote π after deleting every letter larger than k. By construction, $S(\mathcal{F}_{k,\gamma}, \pi) = F_k(\pi) = F_k(\pi^{\leq k})$ if $\pi_1^{-1} < 2\gamma n$ and $S(\mathcal{F}_{k,\gamma}, \pi) = G_k(\pi) = G_k(\pi^{\leq k})$ otherwise. That is, $S(\pi, \mathcal{F}_{k,\gamma})$ depends entirely on $\pi^{\leq k}$ and π_1^{-1}. In particular,

$$\begin{aligned} \mathbb{E}[S(\mathcal{F}_{k,\gamma}, \boldsymbol{\pi})] = &\sum_{t < 2\gamma n} \sum_{\sigma \in \mathfrak{S}_{2,k}} F_k(\sigma) \Pr[\boldsymbol{\pi}_1^{-1} = t \ \cap\ \boldsymbol{\pi}^{\leq k} = \sigma] \\ &+ \sum_{t \geq 2\gamma n} \sum_{\sigma \in \mathfrak{S}_{2,k}} G_k(\sigma) \Pr[\boldsymbol{\pi}_1^{-1} = t \ \cap\ \boldsymbol{\pi}^{\leq k} = \sigma]. \end{aligned} \tag{4}$$

With this in mind, we wish to determine $\Pr[\boldsymbol{\pi}_1^{-1} = t \ \cap\ \boldsymbol{\pi}^{\leq k} = \sigma]$. Fix some $\sigma \in \mathfrak{S}_{2,k}$ and define $q = \sigma_1^{-1} - 1$. We claim that the total number of $\pi \in \mathfrak{S}_{2,n}$ with $\pi_1^{-1} = t$ and $\pi^{\leq k} = \sigma$ is

$$\binom{t-1}{q}\binom{2n-t}{2k-1-q}\frac{(2n-2k)!}{2^{n-k}}. \tag{5}$$

Indeed, first we choose the indices where the letters of type $\{1, \dots, k\}$ will go as follows. The leftmost '1' must go in position t, and then one chooses the q positions to the left of t and the $2k-1-q$ positions to the right of t where the remainder of these symbols go in $\binom{t-1}{q}\binom{2n-t}{2k-1-q}$ ways. Once these indices are chosen, the relative order of the symbols in $\{2, \dots, k\}$ determined by σ. One then arranges the remaining symbols that are not in $\{1, \dots, k\}$ in $(2n-2k)!/2^{n-k}$ total ways. This proves the claim. We note that (5) can be written as

$$\begin{aligned} &\frac{(t-1)(t-2)\cdots(t-q)\cdot(2n-t)\cdots(2n-t-2k+q)\cdot(2n-2k)!}{2^{n-k} q!(2k-1-q)!} \\ &\geq \frac{(t-2k)^q (2n-t-2k)^{2k-1-q}(2n-2k)!}{2^{n-k} q!(2k-1-q)!}, \end{aligned}$$

where we used that $0 \leq q < 2k$. By dividing this by $|\mathfrak{S}_{2,n}| = (2n)!/2^n$, we conclude that

$$\begin{aligned} \Pr[\boldsymbol{\pi}_1^{-1} = t \ \cap\ \boldsymbol{\pi}^{\leq k} = \sigma] &\geq \frac{2^k}{q!(2k-1-q)!} \frac{(t-2k)^q(2n-t-2k)^{2k-1-q}}{(2n)\cdots(2n-2k+1)} \\ &\geq \frac{2^k}{q!(2k-1-q)!} \frac{(t-2k)^q(2n-t-2k)^{2k-1-q}}{(2n)^{2k}}. \end{aligned} \tag{6}$$

Let

$$\phi_{q,n}(x) := \frac{2^k}{2n \cdot q!(2k-1-q)!}\left(\frac{x}{2n}\right)^q \left(1 - \frac{x}{2n}\right)^{2k-1-q}.$$

We claim that (6) is equal to $\phi_{q,n}(t) + O(n^{-2})$ for $1 \leq t \leq 2n$, which implies

$$\Pr[\boldsymbol{\pi}_1^{-1} = t \ \cap\ \boldsymbol{\pi}^{\leq k} = \sigma] \geq \phi_{q,n}(t) + O(n^{-2}). \tag{7}$$

Indeed, consider the numerator of (6) as a polynomial in $2k$. Because $t \leq 2n$, for any integer α the coefficient of $(2k)^\alpha$ in this polynomial will be at most $(2n)^{2k-1-\alpha}$, which is $O(n^{2k-2})$ if $\alpha \geq 1$. As $2k$ is finite, the sum of these $2k-1$ terms in the numerator involving $2k$ is at most $O(n^{2k-2})$. Dividing this by the denominator $(2n)^{2k}$ gives the result.

Define

$$f_{k,n}(x) = \sum_{q=0}^{2k-1} \sum_{\substack{\sigma \in \mathfrak{S}_{2,n}, \\ \sigma_1^{-1}=q+1}} F_k(\sigma)\phi_{q,n}(x),$$

and similarly define $g_{k,n}(x)$ but with $G_k(\sigma)$ used instead of $F_k(\sigma)$. Then (4) and (7) imply

$$\mathbb{E}[S(\mathcal{F}_{k,\gamma}, \boldsymbol{\pi})] \geq \sum_{t<2\gamma n} f_{k,n}(t) + \sum_{t \geq 2\gamma n} g_{k,n}(t) + O(n^{-1}).$$

We wish to replace these sums with integrals using Lemma 5.1. Observe that $|f_{k,n}(t)| \leq \frac{k2^k}{n}|\mathfrak{S}_{2,k}|$ for $0 \leq t \leq 2n$ since $|\phi_{q,n}(t)| \leq \frac{2^{k-1}}{n}$ in this range and $F_k(\sigma) \leq 2k$ for all $\sigma \in \mathfrak{S}_{2,k}$. Further, $f_{k,n}(x)$ has degree at most $2k$, so we can break $\mathbb{R}$ into at most $2k$ intervals which f is monotone on. Similar analysis holds for $g_{k,n}(x)$. By taking $r = 2k = O(1)$ and $M = \frac{2k}{n}|\mathfrak{S}_{2,k}| = O(n^{-1})$, we conclude that

$$\begin{aligned}
\mathbb{E}[S(\mathcal{F}_{k,\gamma}, \boldsymbol{\pi})] &\geq \int_1^{\lfloor \gamma n \rfloor - 1} f_{k,n}(x)dx + \int_{\lceil \gamma n \rceil}^{2n-1} g_{k,n}(x)dx + O(n^{-1}) \\
&= \int_0^{\gamma n} f_{k,n}(x)dx + \int_{\gamma n}^{2n} g_{k,n}(x)dx + O(n^{-1}),
\end{aligned}$$

where we used that $|g_{k,n}(x)|, |f_{k,n}(x)| = O(n^{-1})$ in this range to tweak the limits of integration.

At this point one simply needs to choose some k for which the polynomials $f_{k,n}(x)$ and $g_{k,n}(x)$ are feasible to compute and then to choose γ so that the corresponding integral is optimized. For example, taking $k = 2$ and using (3) gives

$$f_{2,n}(x) = (2+4+3)\frac{1}{3n}(1-x/2n)^3 + (2+3)\frac{1}{n}(x/2n)(1-x/2n)^2 + 1 \cdot \frac{1}{n}(x/2n)^2(1-x/2n),$$

and similarly

$$g_{2,n}(x) = (4+3+2)\frac{1}{3n}(1-x/2n)^3 + (3+2)\frac{1}{n}(x/2n)(1-x/2n)^2 + 2 \cdot \frac{1}{n}(x/2n)^2(1-x/2n).$$

In this case $g_{2,n}(x) \geq f_{2,n}(x)$ for all $x \geq 0$, so the optimal choice is $\gamma = 0$, and evaluating this gives the bound

$$\mathbb{E}[S(\mathcal{F}_{2,0}, \boldsymbol{\pi})] \geq 8/3 + O(n^{-1}).$$

In a similar fashion, it is feasible to compute the polynomials $f_{k,n}(x)$ and $g_{k,n}(x)$ for $k = 6$ by use of a computer. By taking $\gamma = .35$, we obtain the bound

$$\mathbb{E}[S(\mathcal{F}_{6,.35}, \boldsymbol{\pi})] \geq 2.9143 + O(n^{-1}),$$

so for n sufficiently large we obtain our desired bound.

For larger m essentially the exact same proof works, the only significant change being that we use

$$\phi_{q,n}(x) = \frac{(m!)^k}{(mn)q!(mk-1-q)!}(t/mn)^q(1-t/mn)^{mk-1-q}.$$

For $m=3$, if we take $k=5$ and $\gamma=.25$, this method ends up giving the desired lower bound for $\mathbb{E}[\mathbf{S}_{3,n}]$ when n is sufficiently large. □

We close this section with a few comments about this proof method . The first is that this method does not seem to give a significant improvement over the safe strategy if k is close to m. As $|\mathfrak{S}_{m,m+1}|$ grows fairly quickly (it is larger than 10^{11} for $m=4$), this proof technique seems to be impractical for m larger than 3.

A similar argument can be used to bound a (γ,γ')-like strategy to further improve the lower bound of Theorem 1.1, and in principle many other strategies of this flavor could be analyzed using this same technique of introducing a finite cutoff value in order to prove asymptotic bounds.

6 Proof of Theorem 1.3: the Avoiding Strategy $\mathcal{A}$

We recall that the avoiding strategy $\mathcal{A}$ was defined in Section 3 as the strategy of guessing '1', then '2', and so on until some 'k' is guessed correctly, and then 'k' is guessed the rest of the game. To determine $\mathbb{E}[S(\mathcal{A},\boldsymbol{\pi})]$, we recall the Bonerroni inequalities, which are essentially weaker versions of the principle of inclusion-exclusion.

Lemma 6.1 ([14])**.** *Let $A_1,\ldots,A_{k-1}$ be events in a finite probability space. For all even L we have*

$$\Pr\left[\bigcap \overline{A_t}\right] \le \sum_{\ell=0}^{L}(-1)^\ell \sum_{1\le t_1<\cdots<t_\ell<k} \Pr\left[\bigcap_{j=1}^{\ell} A_{t_j}\right],$$

and for all odd L we have

$$\Pr\left[\bigcap \overline{A_t}\right] \ge \sum_{\ell=0}^{L}(-1)^\ell \sum_{1\le t_1<\cdots<t_\ell<k} \Pr\left[\bigcap_{j=1}^{\ell} A_{t_j}\right].$$

We use this to prove our main technical lemma of this section.

Lemma 6.2. *For $\pi\in\mathfrak{S}_{m,n}$, let $f(\pi)$ denote the first index which is guessed correctly using the avoiding strategy $\mathcal{A}$, with $f(\pi):=\infty$ if no such index exists. For m fixed and any $\epsilon>0$, we have for all $k\le mn$,*

$$\left|\Pr[f(\boldsymbol{\pi})=k]-e^{-k/n}n^{-1}\right| \le \epsilon n^{-1}+O_{\epsilon,m}(n^{-2}),$$

and moreover

$$\left|\Pr[f(\boldsymbol{\pi})=k]\cdot\mathbb{E}[S(\mathcal{A},\boldsymbol{\pi})|f(\boldsymbol{\pi})=k]-\left(m-\frac{(m-1)k}{mn}\right)e^{-k/n}n^{-1}\right| \le \epsilon n^{-1}+O_{\epsilon,m}(n^{-2}).$$

Proof. For simplicity we first prove these results for $k \leq n$. For $t \leq n$ let A_t be the event $\boldsymbol{\pi}_t = t$. Also define A_* to be the event $\boldsymbol{\pi}_{k+1} = k$. Observe that $f(\boldsymbol{\pi}) = k$ if and only if A_k occurs and no other A_t with $t < k$ occurs. Given $f(\boldsymbol{\pi}) = k$, the expected score is 1 plus the expected number of k's after index k, which by linearity of expectation is 1 plus $mn - k$ times the probability of $\boldsymbol{\pi}_{k+1} = k$. In total then we conclude

$$\Pr[f(\boldsymbol{\pi}) = k] = \Pr\left[\bigcap_{t<k} \overline{A_t} \cap A_k\right], \tag{8}$$

and similarly

$$\begin{aligned}\Pr[f(\boldsymbol{\pi}) = k] \cdot \mathbb{E}[S(\mathcal{A}, \boldsymbol{\pi})] &= \Pr\left[\bigcap_{t<k} \overline{A_t} \cap A_k\right] \cdot \left(1 + (mn-k)\Pr\left[A_* | \bigcap_{t<k} A_t \cap A_k\right]\right) \\ &= \Pr\left[\bigcap_{t<k} \overline{A_t} \cap A_k\right] + (mn-k)\Pr\left[\bigcap_{t<k} \overline{A_t} \cap A_k \cap A_*\right]. \end{aligned} \tag{9}$$

We wish to bound these quantities using the Bonerroni inequalities. We claim for any $1 \leq t_1 < \cdots < t_\ell \leq k \leq n$ that

$$\Pr[A_{t_1} \cap \cdots \cap A_{t_\ell}] = \frac{(mn-\ell)! m^\ell}{(mn)!}.$$

Indeed, every $\pi \in \mathfrak{S}_{m,n}$ with the property $\pi_{t_j} = t_j$ for $1 \leq j \leq \ell$ can be formed by setting $\pi_{t_j} = t_j$ (with the t_j values all distinct since $k \leq n$) and then the remaining positions can be filled in $(mn-\ell)!(m!)^{-n+\ell}((m-1)!)^{-\ell}$ ways. The total number of $\pi \in \mathfrak{S}_{m,n}$ is $(mn)!(m!)^{-n}$, so dividing these two quantities gives the claim. Using $(mn)^\ell \geq \frac{(mn)!}{(mn-\ell)!} \geq (mn-\ell+1)^\ell$, we in particular find

$$n^{-\ell} \leq \Pr[A_{t_1} \cap \cdots \cap A_{t_\ell} \cap A_k] \leq n^{-\ell-1} + O_\ell(n^{-\ell-2}). \tag{10}$$

Similarly we find

$$\begin{aligned}\Pr\left[\bigcap A_{t_j} \cap A_k \cap A_*\right] &= \frac{(mn-\ell-2)!(m!)^{-n+\ell+1}((m-1)!)^{-\ell}(m-2)!^{-1}}{(mn)!(m!)^{-n}} \\ &= \frac{m-1}{m} n^{-\ell-2} + \Theta_\ell(n^{-\ell-3}). \end{aligned} \tag{11}$$

Using the Bonerroni inequality and (10) gives for any even $L \leq k$ that

$$\begin{aligned}\Pr\left[\bigcap_{t<k} \overline{A_t} \cap A_k\right] &\leq \sum_{\ell=0}^{L} (-1)^\ell \sum_{1 \leq t_1 < \cdots < t_\ell < k} \Pr[\bigcap A_{t_j} \cap A_k] \\ &= n^{-1} \sum_{\ell=0}^{L} \frac{(-k/n)^\ell}{\ell!} + O_\ell(k^\ell n^{-\ell-1}) \\ &= n^{-1} e^{-k/n} + O(L^{-L} n^{-1}) + O_L(n^{-2}), \end{aligned} \tag{12}$$

where this last step used $|e^x - \sum_{\ell=0}^{L} \frac{x^\ell}{\ell!}| \le \frac{e^x}{(L+1)!} = O(L^{-L})$ for $x \le 1$. Similarly one can use (11) to show

$$(mn-k)\Pr\left[\bigcap_{t<k} \overline{A_t} \cap A_k \cap A_*\right] = \left(1 - \frac{k}{mn}\right)(m-1)n^{-1}e^{-k/n} + O(L^{-L}n^{-1}) + O_L(n^{-2}), \quad (13)$$

and using odd L gives the reveres inequalities. We conclude the first result for $k \le n$ by using (8) and (12) after taking L sufficiently large so that $O(L^{-L}n^{-1}) \le \epsilon n^{-1}$. Similarly we conclude the second result for $k \le n$ by using (9), (12), (13), and taking L to be sufficiently large.

We now consider the general case. Given $t \le mn$, let $1 \le t^{(1)} \le n$ and $0 \le t^{(2)} < m$ be the unique integers such that $t = t^{(1)} + t^{(2)}n$. Let A_t be the event $\boldsymbol{\pi}_t = t^{(1)}$ and A_* the event $\boldsymbol{\pi}_{k+1} = k^{(1)}$. With this notation established, the exact same argument as above will work, with the only issue being that the bounds of (10) and (11) will be incorrect whenever we have two entries in our sequence with the same $t^{(1)}$ value. However, the number of such sequences is $O(k^{\ell-1})$, and one can show that the probabilities are still $\Theta_\ell(n^{-\ell-1})$ for any sequence. Thus these terms get absorbed in the error term and the bound continues to hold. □

With this we can determine the asymptotic value of $\mathbb{E}[S(\mathcal{A}, \boldsymbol{\pi})]$.

Proof of Proposition 3.1. Note that $\mathbb{E}[S(\mathcal{A}, \boldsymbol{\pi})] = \sum \Pr[f(\boldsymbol{\pi}) = k]\mathbb{E}[S(\mathcal{A}, \boldsymbol{\pi})|f(\boldsymbol{\pi}) = k]$, where this conditional expectation is 0 for $k = \infty$. Using this, Lemma 6.2, and Lemma 5.1, we find for all $\epsilon > 0$

$$\begin{aligned}\mathbb{E}[S(\mathcal{A}, \boldsymbol{\pi})] &\le n^{-1}\sum_{k=1}^{mn}\left(m - \frac{(m-1)k}{mn}\right)e^{-k/n} + \epsilon + O_{\epsilon,m}(n^{-1}) \\ &= \int_0^m \left(m - \frac{(m-1)x}{m}\right)e^{-x}dx + m\epsilon + O_{\epsilon,m}(n^{-1}) \\ &= m - 1 + m^{-1} - m^{-1}e^{-m} + m\epsilon + O_{\epsilon,m}(n^{-1}).\end{aligned}$$

A similar lower bound holds with ϵ replaced by $-\epsilon$, and by taking ϵ arbitrarily small we conclude the desired result. □

The last fact we need to prove Theorem 1.3 is the following.

Lemma 6.3. *Given a strategy $\mathcal{G}$ with Yes/No feedback, let $X(\mathcal{G})$ be the event that $S(\mathcal{G}, \boldsymbol{\pi}) = 0$. Then $\Pr[X(\mathcal{A})] \ge \Pr[X(\mathcal{G})]$ for all strategies $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be any strategy, and let g_t denote the guess made for $\boldsymbol{\pi}_t$ assuming no correct guesses have been made up to this point. Let $c_i = |\{t : g_t = i\}|$ and let $\mathfrak{S}_{m,n}(c_1, \ldots, c_n)$ denote the number of permutations which have no 1 in the first c_1 positions, then no 2 in the next c_2 positions, and so on. It is not too difficult to see that $\Pr[X(\mathcal{G})] = |\mathfrak{S}_{m,n}(c_1, \ldots, c_n)|/|\mathfrak{S}_{m,n}|$. Intuitively the quantity $|\mathfrak{S}_{m,n}(c_1, \ldots, c_n)|$ is maximized when $c_i = m$ for all i, and one can formally show this by using Theorem 4 of [2].

In total, $\Pr[X(\mathcal{G})]$ is maximized by any strategy which guesses each card type m times whenever it guesses no cards correctly. The result follows since $\mathcal{A}$ is such a strategy. □

Proof of Theorem 1.3. The upper bound follows from Proposition 3.1. For the lower bound, fix some strategy $\mathcal{G}$. Observe that $\mathbb{E}[S(\mathcal{G}, \boldsymbol{\pi})]$ is always at least the probability that some card is guessed correctly using $\mathcal{G}$, so by Lemma 6.3 this quantity is at least $1 - \Pr[X(\mathcal{A})]$. In the language of Lemma 6.2, this is equal to $\Pr[f(\boldsymbol{\pi}) \leq mn]$. By Lemmas 6.2 and 5.1, this quantity is at least

$$\int_0^m e^{-x} dx - \epsilon m - O_{\epsilon,m}(n^{-1}) = 1 - e^{-m} - \epsilon m - O_{\epsilon,m}(n^{-1}).$$

Taking ϵ arbitrarily small gives the desired result. □

7 Concluding Remarks

There are many questions that remain in this area. Perhaps the most embarrassing gap in our knowledge is in understanding the score under the optimal misère strategy with Yes/No feedback. Theorem 1.3 proves that this quantity is bounded away from 0, but there is still a large gap between this and the upper bound of Theorem 1.1. We conjecture the following.

Conjecture 7.1. *If n is sufficiently large in terms of m, then under an optimal misère strategy with Yes/No feedback,*

$$\mathbb{E}[\mathbf{S}_{m,n}] = m - o(m).$$

Observe that Theorem 2.2 shows that the analogous conjecture for the optimal strategy is true.

It would be of significant interest to determine the asymptotic value of $\mathbb{E}[\mathbf{S}_{2,n}]$ under an optimal strategy with Yes/No feedback, but this seems difficult. We claim that one can adapt the methods of [9] to prove $\mathbb{E}[\mathbf{S}_{m,n}] \leq 2.7m$ for any m provided n is sufficiently large in terms of m. In light of Theorem 2.2, we know that the ratio $\mathbb{E}[\mathbf{S}_{m,n}]/m$ tends to 1 as a function of m provided n is sufficiently large in terms of m. Given this and Theorem 2.3, we suspect the following holds.

Conjecture 7.2. *For any m and $\epsilon > 0$, if n is sufficiently large in terms of n, we have under an optimal strategy with Yes/No feedback,*

$$\mathbb{E}[\mathbf{S}_{m,n}] \leq (e - 1)m + \epsilon.$$

We end this paper with a delightful conjecture regarding the nature of optimal strategies which was proposed in [2]: With Yes/No feedback, if for the tth guess, an optimal strategy says to "Guess i" and one receives "No" for feedback, then an optimal strategy says to 'Guess i" for the $t + 1$st guess.

Acknowledgments

The authors would like to thank Steve Butler, Maryanne Gatto, and Xiaoyu He for fruitful discussions.

References

[1] Blackwell, D., Hodges, J. (1957). Design for the control of selection bias. *Ann. Math. Statist.* 28: 449–460. doi.org/10.1214/aoms/1177706973.

[2] Chung, F., Diaconis, P., Graham, R., Mallows C. (1981). On the permanents of complements of the direct sum of identity matrices. *Adv. Appl. Math.* 2(2): 121–137. doi.org/10.1016/0196–8858(81)90001-4.

[3] Chung, F., Diaconis, P., Graham, R. (2019). Permanental generating functions and sequential importance sampling. *Adv. Appl. Math.* doi.org/10.1016/j.aam.2019.05.004.

[4] Ciucu, M. (1998). No-feedback card guessing for dovetail shuffles.*Ann. Appl. Probab.* 8(4): 1251–1269. doi:10.1214/aoap/1028903379.

[5] Diaconis, P. (1978). Statistical problems in ESP research. *Science*. 201(4351): 131–136. doi.org/10.1126/science.663642.

[6] Diaconis, P., Gatto, M., Graham, R. (1980). Suboptimal strategies for a card guessing experiment involving feedback. Bell Laboratories Tech. Memorandum 80–1216 16.

[7] Diaconis, P., Graham, R. (1981) The analysis of sequential experiments with feedback to subjects. *Ann. Statist.* 9(1): 3–23. doi.org/10.1214/aos/1176345329.

[8] Persi, D., Graham, R., Holmes, S. (2001). Statistical problems involving permutations with restricted positions. In: Gunst, M., Klassaen, C., Vaat, A., eds. *State of the art in probability and statistics*. Beachwood, OH: Institute of Mathematical Sciences, pp. 195-222.

[9] Diaconis, P., Graham, R., He, X., Spiro, S. (2019). Card Guessing with Partial Feedback. arXiv preprint arXiv:2010.05059.

[10] Diaconis, P., and Kolesnik, B. (2019). Randomized sequential importance sampling for estimating the number of perfect matchings in bipartite graphs. arXiv preprint arXiv:1907.02333.

[11] Efron, B. (1971). Forcing a sequential experiment to be balanced. *Biometrika* 58(3): 403–417. doi.org/10.2307/2334377.

[12] Ethier, S. (2010). *The doctrine of chances: probabilistic aspects of gambling.* Springer Science & Business Media.

[13] Ethier, S., Levin, A. (2005). On the fundamental theorem of card counting, with application to the game of trente et quarante. *Advances in applied probability.* 37(1): 90–107. doi.org/10.1007/BF01737563.

[14] Florescu, L., Spencer, J. (2014). *Asymptopia.* American Mathematical Society.

[15] Gatto, M. (1978). Statistical experiments involving feedback: An example. Bell Laboratories Tech. Memorandum 78-1212–7, 1978.

[16] Gilpin, L., Bau, D., Yuan, B., Bajwa, A., Specter, M., Kagal, L. (2018). Explaining explanations: An overview of interpretability of machine learning. *IEEE 5th International Conference on data science and advanced analytics (DSAA)*, pp. 80–89.

[17] Liu, P. (2019). Asymptotic analysis of card guessing with feedback. *Discret. Appl. Math.* 288: 270–278. doi.org/10.1016/j.dam.2020.09.005.

[18] Lundberg, S., Lee, S. (2017). A unified approach to interpreting model predictions. *Advances in neural information processing systems*, pp. 4765–4774.

[19] Shevtsova, I. (2011). On the absolute constants in the Berry-Esseen type inequalities for identically distributed summands. arXiv preprint arXiv:1111.6554.