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An integral equation for the identification of causal effects in nonlinear models

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Abstract

When the causal relationship between X and Y is specified by a structural equation, the causal effect of X on Y is the expected rate of change of Y with respect to changes in X, when all other variables are kept fixed. This causal effect is not identifiable from the distribution of (X, Y). We give conditions under which this causal effect is identified as the solution of an integral equation based on the distributions of (X, Z) and (Y,Z), where Z is an instrumental variable.

Introduction

Suppose the causal relation between two real-valued randomly variables X and Y is specified by the structural equation $Y = f(X, U)$, where U represents all other variables that may also affect Y. We assume $f(s, U)$ is smooth in x, and write $Y(x) = f(x, U)$, $Y^{(i)}(x) = \frac{\partial^i}{\partial x^i} f(x, U)$, i=1,2. We regard $\theta(x) = E(Y^{(1)}(x))$ as the causal effect of X on Y. Discussion of this model and its relation to the potential outcome framework for causal inference was given in Wong (2021). Since $Y(x)$ and $Y^{(i)}(x)$ are not directly obtainable from X and Y, $\theta(x)$ is not identifiable from the distribution (X, Y) alone. The method of instrumental variable attempts to identify $\theta(x)$ from the joint distribution of (X, Y, Z) where the instrumental variable Z can affect X through another equation $X = g(Z, V)$. However, identifiability results using instrumental variables are only available under very strong restrictions f and g. These results and related literature had been reviewed in Wong (2021) and will not be repeated.

We consider the following nonlinear, nonparametric causal model:

- $Y = f(X, U),\ \ Y \in R,\ X \in R,\ U \in R^p,\ \ f$ is bounded and smooth in x (1)
- $X = g(Z, V),\ \ Z \in R^q, V \in R^r$ (2)
- $\sup_{x,z} p_z(x) < \infty$, where $p_z(\ )$ denotes the density function of $X(z)$ (3)
- Z is independent of (U, V) (4)

In (1), the condition that f is bounded and smooth in x means that $\sup_u |f(0, u)| < \infty$ and $\sup_u |\frac{\partial^i}{\partial x^i} f(x, u)| < m(x)$ for *i*=1, 2, where $m(\ )$ is a bounded and integrable function. Then, when $x \to \infty$, we have $Y(\infty) = \lim Y(x)$ exists and $E(Y(x)) \to E(Y(\infty))$. Similarly for $Y(-\infty)$. Also, $\theta(x) = E(Y^{(1)}(x))$ is a differentiable function and $\lim \theta(x)$=0 as $x \to \pm\infty$.

For nonlinear f and g, the independence condition (4) is not sufficient for the identification of $\theta(x)$ from the distribution of (X, Y, Z). Under the condition that changes in Y caused by varying X is uncorrelated to changes in X caused by varying Z, conditional on $Z = z$, Wong (2021) showed that the distributions (X,Z) and (Y,Z) identify a related function $\psi(z) = E(Y^{(i1)}(X)|Z=z)$. That paper also demonstrated by examples that sometimes the function $\theta(x)$ can be recovered from the function ψ (z), but did not provide results on the direct identification of $\theta(x)$. To fill this gap, in this paper we derive an integral equation that can be used to identify $\theta(x)$ from the distributions of (X,Z) and (Y,Z).

Result

To formulate our main result, consider the following conditions:

- $I(X(z) \le x)$ is uncorrelated with $Y^{(1)}(x)$, for all x, z (5)
- The set of distributions of $X|Z=z$, induced by varying z, is a complete set (6)

Theorem: If (1)-(6) hold, then θ is identifiable via the integral equation,

$$\int K(z,x)\theta(x)dx = \mu(z) - \mu(0) \tag{7}$$

$$\text{where} \quad K(z,x) = P(X \le x|Z=0) - P(X \le x|Z=z)$$

$$\mu(z) = E(Y|Z=z)$$

Proof:

$$\mu(z) = E(Y|Z=z) = E(f(X,U)|Z=z) = E(f(g(z,V),U|Z=z)$$
$$= E(Y(X(z)))$$
$$= E\int \delta\big(x - X(z)\big)Y(x)dx \tag{8}$$

Replacing the delta function $\delta(\,)$ by the $N(0,\sigma^2)$ density $\phi_\sigma(\,)$, we define

$$\mu_\sigma(z) = E\int \phi_\sigma\left(x - X(z)\right)Y(x)dx \tag{9}$$

Since $Y(x) = Y(X(z)) + Y^{(1)}(X(z))\big(x - X(z)\big) + \frac{1}{2}Y^{(2)}(X(W))\big(x - X(z)\big)^2$, where W is an intermediate variable lying between x and $X(z)$, hence

$$\mu_\sigma(z) = EY(X(z) + E[\tfrac{1}{2}Y^{(2)}(X(W))\int \phi_\sigma(\big(x - X(z)\big)\big(x - X(z)\big)^2 dx$$
$$= \mu(z) + \frac{\sigma^2}{2}\sup_x m(x)$$

Thus, $|\mu_\sigma(z) - \mu(z)| \le c\sigma^2$ for some constant *c* (10)

Next, we claim that

$$|E\left(\Phi\left(\frac{x-X(z)}{\sigma}\right)Y^{(1)}(x)\right) - P(X(z) \le x)\theta(x)| \le cm(x)\sqrt{\sigma} \text{ for some constant } c \tag{11}$$

Assuming (11) is true, we now analyze the integral in (9). Using integration by part, we have

$$\mu_\sigma(z) = E[Y(\infty) - \int\left(\Phi\left(\frac{x-X(z)}{\sigma}\right)Y^{(1)}(x)\right)dx]$$
$$= E(Y(\infty)) - \int P(X(z) \le x)\theta(x)dx + r(z,\sigma)$$

where for some constant c, $|r(z,\sigma)| \le c\sqrt{\sigma}$ for all small σ.

Thus $|(\mu_\sigma(z) - \mu_\sigma(0)) - \int[P(X(0) \le x) - P(X(z) \le x)]\theta(x)dx| \le 2c\sqrt{\sigma}$ (12)

Taking the limit of (10) and (12) as σ→ 0, we have

$$\mu(z) - \mu(0) = \lim_{\sigma\to 0}(\mu_\sigma(z) - \mu_\sigma(0)) = \int[P(X(0) \le x) - P(X(z) \le x)]\theta(x)dx.$$

The desired equation (7) follows because $P(X(z) \le x) = P(g(z,V) \le x) = P(g(z,V) \le x|Z = z) = P(g(Z,V) \le x|Z = z) = P(X \le x|Z = z)$.

To prove the claim (11),

$$|E\left(\Phi\left(\tfrac{x-X(z)}{\sigma}\right)Y^{(1)}(x)\right) - P(X(z) \le x)\theta(x)|$$

$$= |E\left(\Phi\left(\tfrac{x-X(z)}{\sigma}\right)Y^{(1)}(x)\right) - E(I(X(z) \le x))E(Y^{(1)}(x))|$$

$$= |E\left(\Phi\left(\tfrac{x-X(z)}{\sigma}\right)Y^{(1)}(x)\right) - E(I(X(z) \le x)Y^{(1)}(x))| \quad \text{(by condition (5))}$$

$$\le m(x)\, E|\Phi\left(\tfrac{x-X(z)}{\sigma}\right) - I(X(z) \le x)|$$

$$\le m(x)\, [\Phi\left(-\tfrac{1}{\sqrt{\sigma}}\right) + 4(\sup_{x,z} p_z(x))\sqrt{\sigma}\,] \qquad (13)$$

The last inequality (13) holds because $|\Phi\left(\frac{x-X(z)}{\sigma}\right) - I(X(z) \le x)|$ is bounded by 2 on $A(\sigma)$ and by $\Phi(-\frac{1}{\sqrt{\sigma}})$ on $A(\sigma)^C$, where $A(\sigma)$ is the event $\{|X(z) - x| \le \sqrt{\sigma}\}$.

Since both $K(z,x)$ and $\mu(z)$ in the integral equation (7) are determined by the distributions of (X,Z) and (Y,Z), it follows that θ is also determined if the solution to (7) is unique.

To establish uniqueness, let a be a fixed constant, and define for any $\theta(\ )$, its anti-derivative $\lambda(x) = a - \int_x^\infty \theta(t)dt$. Suppose θ_1 and θ_2 are two solutions to (7) and λ_1 and λ_2 are the corresponding anti-derivatives, then

$$E(\lambda_1(X) - \lambda_2(X)|Z = z) = \int p_{X|Z}(x|z)\,(\lambda_1 - \lambda_2)(x)dx$$
$$= -\int P(X \le x|Z = z)(\theta_1 - \theta_2)(x)dx = -\int P(X \le x|Z = 0)(\theta_1 - \theta_2)(x)dx.$$

Since the last expression does not depend on z, condition (6) implies $\lambda_1 = \lambda_2$, and therefore $\theta_1 = \theta_2$.

Discussion

Of the 6 conditions in the theorem, the first 3 are needed just set up the model and are not restrictive. On the other hand, conditions (4), (5), (6) each represents a significant constraint on the model. Condition (4) says that Z is independent of all other causal variables that affect X and Y. Together with (1) and (2), this means that the only way Z can affect Y causally is indirectly through its effect on X. This seems to be a natural condition on an instrumental variable. Condition (6) implies that the family of conditional distributions $P(X|Z=z)$ as z varies, is a large family. This means that Z has non-trivial relationship with X in the sense that varying the value of z leads to rich changes in the distribution of X. This is also a reasonable condition on an instrumental variable. This type of completeness condition was first introduced into causal inference by Imbens and Newey (2003). Finally, condition (5) requires $Y^{(1)}(x) = \frac{\partial f}{\partial x}(x,U)$ to be uncorrelated to $I(X(z) \leq x) = I(g(z,V) \leq x)$, which is a non-trivial condition not easy to interpret, but is needed to establish the relationship (7) between $\mu(z)$ and $\theta(x)$. Wong (2021) introduced a similar condition that requires $\frac{\partial f}{\partial x}(X,U)$ to be conditionally uncorrelated to $\frac{\partial g}{\partial z}(Z,V)$ given $Z=z$. However, under that condition one can only relate $\mu(z)$ to $\psi(z) = E(\frac{\partial f}{\partial x}(X(z),U))$ but not to $\theta(x) = E(\frac{\partial f}{\partial x}(x,U))$. In the general context of (1)-(4), we are not aware of alternative conditions that be used to relate $\mu(z)$ to $\theta(x)$.

Example: Suppose $Y = U_1 h(X) + U_2$, $X = g(Z,V)$, where $h(\,)$ is a smooth and bounded function in x. If $U_1$is independent of V, then condition (5) is satisfied. In this example, the "subject-level" causal effect $Y^{(1)}(x)$ is assumed to be proportional to a nonlinear function $h(x)$, but heterogeneity is allowed by letting the proportionality constant depend on the subject. On the other hand, no restriction is imposed on the relation between Z and X beyond the completeness condition (6), which is not too restrictive. For example, (6) holds in the following cases (a) $g(z,v) = s(z+v)$ where $s(\,)$ is an invertible function and V is a continuous random variable, (b) $g(z,v) = 1{+}v_1 z + v_2 z^2$, V_1 and V_2 are independent random variables. This example demonstrated the usefulness of our result in a nonlinear, nonparametric model that allows heterogeneity in the causal effect of X on Y in different subjects.

The above proof of the theorem follows the way we discovered the integral equation originally, namely, start with the expression for $E(Y|Z=z)$, replace the delta function in the expression by the normal kernel and then integrate by part to obtain an expression involving $\theta(\,)$. Weijie Su (personal communication) suggests a second proof, which starts from the given $K(z,x)$ and then shows that the integral in (7) gives rise to $\mu(z) - \mu(0)$. His proof has the advantage that it does not require the existence of bounded second derivatives. See Su (2021, arXiv).

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